The distance function from a real algebraic variety.

Nonlinear Algebra in Applications

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The distance function

Let $X \subset \mathbb{R}^n$ be an algebraic variety. The distance function from $u \in \mathbb{R}^n$ to X is

$$d_X(u) := \min_{x \in X} |u - x| \quad \textit{Euclidean Distance.}$$

Basic question: Given X, $d_X(u) = ?$, level sets ?, properties,...

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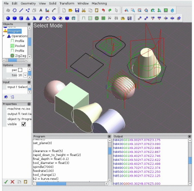
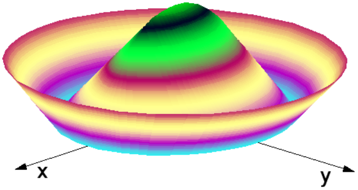
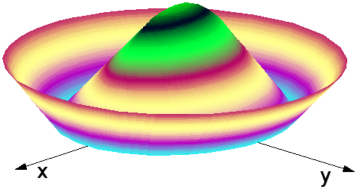
In real world, level sets are offset varieties, they have striking engineering applications, in CAD/CAM manifacturing tools.



Graphic of the distance function from a ellipse



(courtesy by Antonio Greco)



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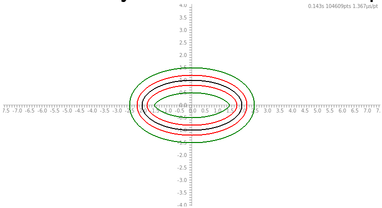
Level curves of this "sombrero" are real octics, with two connected components.

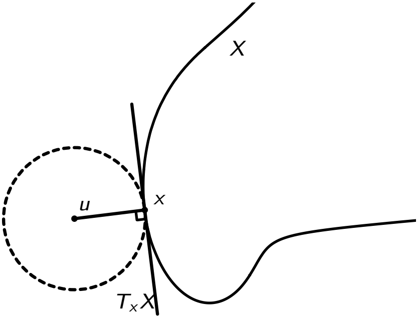
By moving P across the singular set, the point on the ellipse which realizes the minimum distance from P jumps. It remains well defined, by continuation, as a point whose normal meets P.



The critical points of the distance function from u on X

If X is a smooth subvariety, the minimum of the distance from u is attained among the points x such that $T_xX \perp (u - x)$. These are the critical points on X of the distance function from u.

Checking all of the critical points guarantees to compute the global distance from u to X.



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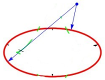

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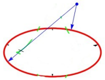

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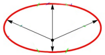

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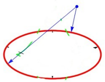

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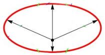

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In the first case there are other 2 complex conjugates normal lines, 4 is the expected and general value (EDdegree(ellipse) $= 4$).

The distance function is an algebraic function

The best way to describe the distance function is as an algebraic function ("a function with multiple values"). In the case of the ellipse E we get $P(u_1, u_2, t) = \sum_{i=0}^{4} p_i(u_1, u_2)t^{2i}$.

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1. For any fixed (u_1, u_2) the roots of $P(u_1, u_2, t) = 0$ are the signed distances from (u_1, u_2) to the critical points. The smallest positive real root is the distance from (u_1, u_2) to E.

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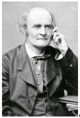
Definition (Draisma-Horobeț-O-Sturmfels-Thomas, O-Sodomaco)

We call $\mathrm{EDpoly}_{X,u}(t^2) = P(u, t)$ the polynomial with the above properties corresponding to a variety X. Its degree is denoted $\mathrm{EDdegree}(X)$ and it counts the number of critical points of the distance function to X from a general $u \in \mathbb{R}^n$.

Steps to compute the ED polynomial with a *Computer Algebra System* like Macaulay2, Singular, CoCoA, Sage,...

Let $q(x) = \sum_{i=0}^{n} x_i^2$ be the Euclidean quadratical form.

1. Pick the ring $\mathbb{Q}[u_0, \ldots, u_n, x_0, \ldots, x_n, t]$
2. Input is the ideal I_X with generators $f = (f_1, \ldots, f_m)$
3. Let $c = \mathrm{codim}\ X$
4. Compute $I_{X_{sing}}$ singular locus, by c-minors of $Jac(f)$.
5. Compute the critical ideal as
$$I_u := \left(I_X + (c+1)\text{-minors of } \begin{pmatrix} u - x \\ Jac(f) \end{pmatrix} \right) : \left(I_{X_{sing}}\right)^{\infty}$$
6. Eliminate $x_0, \ldots, x_n$ in $I_u + \left(t^2 - q(x-u)\right)$, get $\mathrm{EDpoly}_{X,u}(t^2)$.


Arthur Cayley

Cayley computed in XIX century the EDpolynomial of conics, by using invariant theory. His result is

Theorem (Cayley)

- $\mathrm{EDdegree}(E) = 2 \iff$ *E is a circle.*
- $\mathrm{EDdegree}(E) = 3 \iff$ *E is a parabola.*
- $\mathrm{EDdegree}(E) = 4$ *for all other smooth conics.*

Cayley even computed the discriminant of EDpoly.

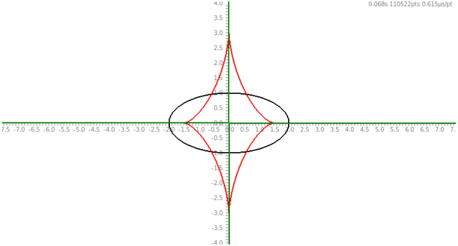
The discriminant of the ED polynomial of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$ is

$$L^3 x^2 y^2,$$

where $c^2 = a^2 - b^2$, L is the evolute with equation the Lamé sextic

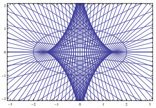
$$L = (a^2x^2 + b^2y^2 - c^4)^3 + 27a^2b^2c^4x^2y^2$$

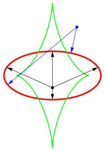
Note the two symmetry axis x, y appear in the discriminant.



This is another general phenomenon, the ED polynomial contains informations on the symmetry axis.

Envelope of the normals, the evolute





The evolute of the ellipse is the Lamé sextic in green. The number of real normals is 4 inside the green curve, is 2 outside the green curve.

Projective varieties

For projective varieties we compute their EDpoly and $\mathrm{EDdegree}$ from their cone.

Example

For the projective ellipse with equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-z^2$$

we consider the distance function induced by the quadratic form $x^2+y^2+z^2$. Recall to saturate the vertex of the cone (x,y,z).

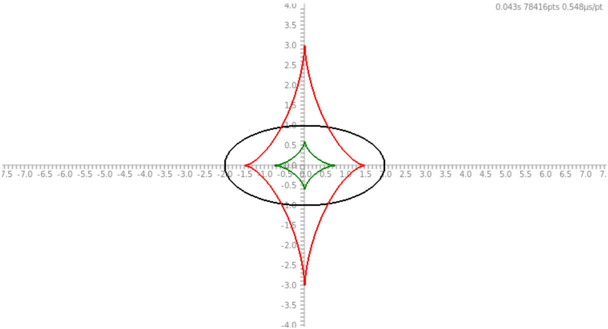
The discriminant of EDpoly now contains L^3 where

$$L=\left(a^2(b^2+1)^2x^2+b^2(a^2+1)^2y^2-c^4z^2\right)^3+27a^2b^2c^4(a^2+1)^2(b^2+1)^2x^2y^2z^2$$

a different Lamé sextic.

Picture of the evolute

The evolute (EDdiscriminant) of the ellipse $x^2 + 4y^2 - 4$, in the affine and in the projective case (setting $z = 1$).

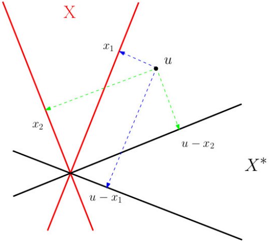


Theorem (Draisma-Horobeţ-O-Sturmfels-Thomas, O-Sodomaco)

Let X be a projective variety and $X^\vee$ its dual. Let $q(u)$ be the Euclidean quadratic form. Then for any data point $u \in V$

1. $\mathrm{EDpoly}_{X,u}(t^2) = \mathrm{EDpoly}_{X^\vee,u}(q(u) - t^2)$.
2. $\mathrm{EDdegree}(X) = \mathrm{EDdegree}(X^\vee)$

The Theorem means that projective duality corresponds to variable reflection for the ED polynomial.



Duality for the ellipse

Let X be the projective ellipse with equation $x^2+4y^2-4z^2=0$.
Then $\mathrm{EDpoly}_{X,(x,y,z)}(t^2) =$
$(x^2+4y^2-4z^2)^2(4x^4+20x^2y^2+25y^4-12x^2z^2+30y^2z^2+9z^4)+$
$(\ldots)t^2+(\ldots)t^4+(4x^2-55y^2-39z^2)(60)t^6+900t^8$.

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$\mathrm{EDpoly}_{X^\vee,(x,y,z)}(t^2) =$
$(4x^2 + y^2 - z^2)^2(64x^4 + 80x^2y^2 + 25y^4 + 48x^2z^2 - 30y^2z^2 + 9z^4) +$
$(\ldots)t^2 + (\ldots)t^4 + (64x^2 + 5y^2 + 21z^2)(-60)t^6 + 900t^8$.

Note in red the equation of the dual ellipse.

Corank one matrices, SVD

There is a significant case where the ED polynomial has a nice form. In the space of $n \times m$ matrices equipped with the L_2-norm $q(A) = \mathrm{tr}(AA^t)$, let X = variety of corank one matrices.

Here the ED polynomial is

$$\mathrm{EDpoly}_{X,A}(t^2) = \det(AA^t - t^2 I), \text{ with roots } \pm\sigma_1, \ldots, \pm\sigma_n$$

For general matrices A of size $n \times m$, with $n \leq m$, there are n critical points $\sigma_i v_i \otimes w_i$ (singular pairs) of the distance function to the variety of rank one matrices.

$A = \sum_i \sigma_i v_i \otimes w_i^t$ is the *Singular Value Decomposition* (SVD) of A.

If A is a symmetric matrix, we get the splitting

$$\det(AA^t - t^2 I) = \det(A - tI)\det(A + tI),$$

the critical points are $v_i v_i^t$ where v_i are eigenvectors of A. We get the spectral decomposition

$$A = \sum_i \lambda_i v_i \otimes v_i^t$$

where λ_i are the eigenvalues of A.

The distance function in a tensor space

Let V_i be real vector spaces equipped with a scalar product $q_i\colon V_i \times V_i \to \mathbb{R}$, equivalently a positive definite quadratic form $q_i\colon V_i \to \mathbb{R}$.
Example: $V_i \simeq \mathbb{R}^{n_i}$ with $q_i(x) = \sum x_i^2$.

The forms q_i altogether define a quadratic form on $V_1 \otimes \ldots \otimes V_d$ by $q(v_1 \otimes \ldots \otimes v_d) := q_1(v_1) \cdots q_d(v_d)$, then extended by linearity.
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A tensor t is *isotropic* if $q(t) = 0$, they fill the isotropic quadric Q.

The singular d-ples

Any tensor $t \in \mathbb{R}^{m_1} \otimes \ldots \otimes \mathbb{R}^{m_d}$ defines a distance function $f_t \colon X = \mathbb{P}^{m_1 - 1} \times \ldots \times \mathbb{P}^{m_d - 1} \to \mathbb{R}$ over the Segre variety X of decomposable tensors.

Theorem (Lim, Qi)

The critical points of f_t corresponds to tensors $(x_1, \ldots, x_d) \in X$ such that

$$t(x_1, \ldots, \hat{x}_i, \ldots, x_d) = \lambda_i x_i$$

which are called singular d-ples.

Reference book: Qi, Luo, "Tensor Analysis and Spectral Theory", SIAM, 2017.

The number of singular d-ples

Theorem (Friedland-O, EDdegree of Segre variety)

The number of singular d-ples of a general tensor t over $\mathbb{C}$ of format $m_1 \times \ldots \times m_d$ is the coefficient of $\prod_{i=1}^{d} t_i^{m_i-1}$ in the polynomial

$$\prod_{i=1}^{d} \frac{\hat{t}_i^{m_i} - t_i^{m_i}}{\hat{t}_i - t_i}$$

where $\hat{t}_i = \sum_{j \neq i} t_j$. This number is $\mathrm{EDdegree}(\mathbb{P}^{m_1-1} \times \ldots \times \mathbb{P}^{m_d-1})$

Theorem (Special case of binary tensors)

$$\mathrm{EDdegree}(\underbrace{\mathbb{P}^1 \times \ldots \times \mathbb{P}^1}_{d}) = d!$$

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Question: are there simpler ways to express these numbers ?

The generating function and a first asymptotics

Theorem (Zeilberger)

Let $a_d(k_1, \ldots, k_d)$ be the number of critical points of format $\prod_{i=1}^{d}(k_i + 1)$ then

$$\sum_{k \in \mathbb{N}^d} a_d(k_1, \ldots, k_d)\mathbf{x}^k = \frac{1}{\left(1 - \sum_{i=2}^{d}(i-1)e_i(\mathbf{x})\right)} \prod_{i=1}^{d} \frac{x_i}{1 - x_i}$$

where e_i is the i-th elementary symmetric function.

Theorem (Zeilberger, Pantone)

$$a_3(n,n,n) \sim \frac{2}{\sqrt{3\pi}} \frac{8^n}{n} \qquad \textit{for } n \to \infty$$

Tensor Eigenvectors in the symmetric case

The critical points of the distance function from a symmetric tensor $A \in \mathrm{Sym}^d V$ to the Veronese variety have the form λv^d, v such that $q(v) = 1$ is eigenvector with eigenvalue λ.

Theorem (Fornaess-Sibony, Cartwright-Sturmfels)

The number of eigenvectors of a symmetric tensor $A \in \mathrm{Sym}^d \mathbb{C}^m$ is (for $d \geq 2$)

$$\frac{(d-1)^m - 1}{d-2}$$

This number is $\mathrm{EDdegree}$ of d-Veronese embedding of $\mathbb{P}^{m-1}$.

Theorem (Qi)

If X = discriminant hypersurface, and d is even,

$$\mathrm{EDpoly}_{X,f}(t^2) = \Delta_d\left(f(x) - tq(x)^{d/2}\right)\Delta_d\left(f(x) + tq(x)^{d/2}\right).$$

Catanese-Trifogli formula

Let X smooth projective, $\dim X = m$

Theorem (Catanese-Trifogli)

If X is transversal to Q then

$$\mathrm{EDdegree}(X) = \sum_{i=0}^{m} (-1)^i (2^{m+1-i} - 1) c_i(X)$$

where c_i are Chern classes.

If X is affine, transversality is needed with both the hyperplane at infinity and the quadric at infinity. This explains the different behaviour proved by Cayley concerning circle, parabola and general conic.

The singular case

Theorem (Piene, Aluffi)

Let $X \subset \mathbb{P}^N$ possibly singular. Assume that Q is transversal to a Whitney stratification of X. Then the same formula holds

$$\mathrm{EDdegree}(X) = \sum_{i=0}^{m} (-1)^i (2^{m+1-i} - 1) c_i^M(X)$$

where c_i^M are Chern-Mather classes.

Crash course on Chern-Mather classes

Let $X \dashrightarrow Gr(\mathbb{P}^m, \mathbb{P}^N)$ be the Gauss map defined on smooth points. Consider the closure of the graph in $X \times Gr(\mathbb{P}^m, \mathbb{P}^N)$, with its projection to X is the Nash blow-up $\tilde{X}$ of X.

Consider the Chern classes of the universal bundle, pullback them to $\tilde{X}$. Their push-forward to X are the Chern-Mather classes (in the Chow ring of X).

If X is smooth the Gauss map is defined everywhere and we have $c_i(X) = c_i^M(X)$.

The highest term of ED polynomial

Proposition (O-Sodomaco)

Let $X \subset \mathbb{P}(V)$, possibly singular, be transversal to the isotropic quadric Q, then

$$\mathrm{EDpoly}_{X,u}(t^2) = \sum_{i=0}^{d} p_i(u) t^{2i},$$

where $d = \mathrm{EDdegree}(X)$ and $p_i(u)$ is homogeneous of degree $2d - 2i$. In particular the ED polynomial of X is monic.

The transversality assumption implies that the distance function is a integral element in the algebraic ring extension.

The transversality assumption is necessary in the previous Theorem, as it is shown by

Theorem (Sodomaco)

Notations as above, let X be the Veronese variety $v_d(\mathbb{P}^n)$. Then

$$p_0(u) = Disc^2(u) \qquad p_{max}(u) = \Delta_{\tilde{Q}}^{d-2}(u)$$

where $\tilde{Q}$ is the d-Veronese embedding of Q and $\Delta_{\tilde{Q}}$ is its dual.

A Corollary is the rational formula $\pm\frac{Disc(u)}{\Delta_{\tilde{Q}}^{(d-2)/2}(u)}$ for the product of eigenvalues, generalizing to tensors the fact that product of eigenvalues of a matrix is the determinant.

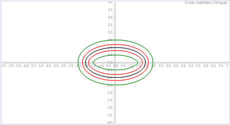
Theorem

Let $X \subset \mathbb{P}(V)$ be irreducible, possibly singular, suppose that X and $X^\vee$ are transversal to Q. Let $u \in V$ and g be the equation of $(X^\vee \cap Q)^\vee$.

1. *If* $\mathrm{codim}(X) \geq 2$, *then*

$$\mathrm{EDpoly}_{X,u}(0) = g.$$

2. *If X is a hypersurface, then*



$$\mathrm{EDpoly}_{X,u}(0) = f^2 g$$

where f is the equation of X.

Thanks

Thanks for your attention !!